Coming up Short: Managing Underfunded Portfolios in a LDI-ES Framework[1]

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Abstract

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We employ a liability directed investment (LDI) rebalancing framework based on expected shortfall (ES), which we refer to as LDI-ES, to prescribe remedies for an underfunded portfolio. Investors in the LDI-ES framework face a risky asset, such as a stock index, and a risk-free bond. They begin with some level of current wealth and set their target wealth at the end of N periods, and their tolerance for shortfalls from that target wealth. Portfolio rebalancing in the LDI-ES framework is contrasted with common fixed-proportions rebalancing, where portfolio allocations are rebalanced to ratios, such as 60:40, at the beginning of each of N periods. We consider critical issues of underfunding, where there is no portfolio that can meet the shortfall constraint, and we explore the effectiveness of (a) portfolio infusions in resolving underfunded situations, relative to other measures such as (b) increasing risk, (c) cutting back on target liabilities/goals, and (d) extending portfolio horizon.

Keywords: Liability-directed-investing; behavioral portfolio theory; expected shortfall; rebalancing; infusions.

JEL codes: G02; G11; G23.

1 Introduction

1.1 Overview

Behavioral portfolio theory, developed in Shefrin and Statman (2000), describes goal-driven investors, seeking to maximize expected wealth, yet concerned about the risk of shortfalls from their targets. The setting of Shefrin and Statman's model is a single-period setting, calling for the construction of an optimal portfolio that is not rebalanced over time when it becomes non-optimal. We explore portfolio rebalancing in an LDI-ES framework where investors aim to maximize expected portfolio value while minimizing expected shortfall, and we analyze the reconstruction of underfunded portfolios in a multi-period setting.

Consider an optimal initial portfolio composed of some proportion in a risky asset, such as a stock index, and a risk-free bond. That portfolio provides the highest expected terminal wealth at the end of N periods, subject to a constraint on the amount of expected shortfall from a target level of terminal wealth. We can think of the target level of terminal wealth as the minimum amount that an individual investor must have for retirement income or the minimum amount a pension fund must have to satisfy liabilities.

The optimal initial portfolio is not likely to be optimal at the beginning of the second period, because the factors determining the optimal portfolio have likely changed during the period. The wealth of our investor at the beginning of the second period may be higher or lower than her wealth at the beginning of the initial period, and our investor now faces $(N-1)$ periods before the terminal date. Rebalancing at the beginning of the second period consists of changes in portfolio allocations from the optimal allocation at the beginning of the initial period to the optimal allocation at the beginning of the second period. We refer to this static repeated rebalancing method as the LDI-ES rebalancing method. In contrast, in the fixed proportions (FP) rebalancing method, allocations at the beginning of the second period are rebalanced to some fixed proportions, perhaps the proportions at the beginning of the initial period or 60-40 proportions where 60 percent is allocated to the risky asset and 40 percent to the risk-free one.

There will be times when our investor finds at the beginning of any period that her wealth declined so much during the previous period she cannot construct any portfolio that conforms to the desired target terminal wealth and shortfall allowance. Pension plans often describe these situations in the language of *underfunding*.[1]

We consider four ways in which pension funds can remedy underfunding. First, by cash infusions from their sponsors that increase current wealth. Second, pension funds can also remedy underfunding by reductions of desired target terminal wealth, i.e., entitlements, by directly reducing promised benefits to fund beneficiaries. Third, increasing shortfall tolerance, i.e., taking on more risk. And fourth, postponing the terminal date to give more time to reach portfolio targets. Individuals in the same circumstances can infuse cash into their retirement portfolios by saving more and spending less, reducing their desired target retirement wealth, increasing their shortfall tolerance, or postponing their retirement date. Comparing these alternate paths to redressing underfunding is the focus of this paper.

[1] Traditional underfunding is defined in terms of the present value of liabilities exceeding the current value of assets in the pension plan. Whereas this definition depends critically on the discount rate used to present value liabilities, the definition of underfunding in our LDI-ES framework is endogenous to the optimization itself, and bypasses the debate around discount rates.

1.2 Extant literature

Related work includes Binsbergen and Brandt (2009), who show that in a dynamic setting with shortfall constraints, ex-post risk controls offer greater utility gains than ex-ante measures. A related dynamic optimization literature examines the optimal way to beat a benchmark or minimize the time to reach a benchmark (see Basak, Shapiro, and Tepla (2006) and Browne (2000)). These papers deal mostly with optimization, whereas our paper focuses on rebalancing and underfunding.

For long-term liability matching, it is also recognized that risk in portfolio management cannot be managed through diversification alone. Shortfall and underfunding are relevant risks and require a multi-period portfolio construction approach where shortfall risks are given due consideration and optimization accounting for this metric is imposed, as discussed in Amenc, Martellini, Goltz, and Milhau (2010). Also, liabilities are no longer static, and interest rate and inflation risk also matter. Early papers on portfolio construction for liability matching include those of Sharpe and Tint (1990) and Martellini and Milhau (2009).

Ang, Chen, and Sundaresan (2013) further develop the model of Sharpe and Tint (1990), representing the objective function as a natural construction containing an exchange option between the assets and liabilities. They find that the weight in the risky asset follows a U-shaped path as we go from being deeply underfunded to fully funded. When the funding level is very low, the manager is forced to assume a high level of risk, placing greater weight on the risky asset in the portfolio. As the portfolio reaches a level close to being funded, the weight on the risky asset is reduced so as not to jeopardize meeting the shortfall constraint. Once the portfolio has reached a healthy level of funding, the risky asset position is ramped up again. Their paper, though related to underfunding, does not address specific alternative remedies, which is the focus of this paper.

In the context of expected shortfall, there is also increasing recognition that higher-order moments also matter and require better methods for estimation, as discussed in Martellini and Ziemann (2010). Some of these risks of shortfall may be managed in a static model using options, as demonstrated in Das and Statman (2013). Several other papers have explored the static problem with shortfall constraints or VaR (value-at-risk) constraints, such as Roy (1952), Levy and Sarnat (1972), Basak and Shapiro (2001), Basak, Shapiro, and Tepla (2006), and Akcay and Yalcin (2010).

Despite these advances, the literature thus far has not considered multi-period rebalancing in which underfunding is considered directly. We introduce a novel rebalancing approach, within which we assess alternative remedies to underfunding, and we compare these to fixed-proportion rebalancing. In this study we abstract away from the stochastic nature of liabilities[2] (and the discount rates applied to value them) in order to focus more sharply on shortfall risks and the relative effectiveness of various prescriptions in underfunded situations.

[2]Of course, this issue of liabilities is non-trivial. Novy-Marx and Rauh (2011) calculate the present value of state employee pension liabilities as of June 2009 using discount rates that reflect taxpayer risk with estimates ranging from $3.2 trillion to $4.4 trillion. Overall, the liabilities are huge, and the exact amount is subject to contentious debate.

1.3 Overview of results

Some of our results are intuitive; others are not. Here is a brief summary of the findings of this paper.

1. *Portfolio allocations*: We find that investors with greater shortfall tolerance allocate more to the risky asset than investors with lower shortfall tolerance, and that greater shortfall tolerance brings greater expected terminal wealth. Moreover, in most cases, greater portfolio values induce greater allocation to risky assets as there is a smaller chance that additional risk will result in a shortfall, while targeting a higher mean wealth at the horizon.

2. *LDI-ES rebalancing versus fixed-proportion rebalancing*: In LDI-ES, for each period, the risky asset weight is rebalanced to the one that, based on current portfolio wealth and remaining time to horizon, maximizes expected terminal wealth subject to the expected shortfall constraint, allowing for infusions (and alternate remedies) when underfunding occurs. In the first period, the initial portfolio weights are determined, and for comparison, we contrast the terminal wealth outcomes of the LDI-ES scheme with a scheme in which initial weights are held constant, i.e., rebalancing occurs to a fixed proportion (FP) scheme with the initial weights. We will see that when the investor has a high cost of shortfall and a stringent shortfall constraint, he is better off under FP than LDI-ES, but otherwise, he is usually better off under LDI-ES. Furthermore, the (positive) skewness of terminal wealth with LDI-ES is much higher than that under the FP scheme, meaning that it is a dominating strategy for investors with skewness preference.

3. *Role of infusions*: We analyze infusions to an underfunded portfolio and observe some regularities.[3] First, required infusions tend to get larger as we approach the portfolio horizon. Intuitively, if the portfolio is off course with many years left to the horizon, there is time to correct the problem, but when there is little time left, more drastic remedies are needed and infusions are higher. Second, the distribution of infusions is right-skewed, i.e., there are some large outliers that make the worst case underfunding scenarios costly to correct. Third, for investors that have high shortfall risk aversion (i.e., the allowable shortfall is low), expected infusions are naturally high. As the allowable shortfall is relaxed, the expected infusions increase, since these investors shift their portfolios to have greater weight on the risky asset. However, for investors that have high shortfall risk tolerance (i.e., the allowable shortfall is very high), expected infusions are small, as the investor is willing to live with high levels of underfunding. Therefore, expected infusions follow an inverted u-shape as risk tolerance increases in the ES sense.

4. *Remediation of underfunded portfolios*: We also explore three remedial alternatives to making infusions, such as extending the horizon of the portfolio, taking on more shortfall risk, or reducing target liabilities/goals. For comparison we look at the mean and variance of terminal wealth, the probability of shortfall, and ex-post realized average

[3] When the portfolio is geared to target a specific goal amount, rather than a specific date or a given risk tolerance, then infusions play a key role as in the SmartNest algorithm of DFA. See "DFA Gives Managed Accounts a New Dimension," in *Retirement Income Journal*, by Kerry Pechter, October 24, 2012. DFA calls this infusion dependent portfolio targeting "Managed DC".

> shortfall, across all remedies. We find that in the mean-variance metric all the four approaches to dealing with underfunding appear to be quite similar, but when considering a utility function that depends also on higher-order moments over expected terminal wealth and expected shortfall, the infusion approach is almost always dominated by the other three approaches, unless the investor is risk-seeking. Hence, we conclude that infusions are usually not optimal solutions to underfunded portfolios, and changing goals, either in terms of time horizon or entitlements, is better, as is taking on more risk. The classic case of underfunded portfolios is that of social security systems in several countries. Some countries have debated cutting back on entitlements such as in the United States. Other countries decided to take on more risk in their social security investments, for example Chile. Our analyses suggest that these solutions may be more palatable than raising taxes to fund social security or diverting resources from other national projects.[4]

The rest of the paper proceeds as follows. In Section 2 we present the notation and mathematics for LDI-ES rebalancing. Section 3 presents the results of the rebalancing under LDI-ES, and Section 4 delivers an analysis of remedies for underfunded portfolios. This section also introduces the metric under which the various remedies are compared, and presents the features of terminal wealth distributions. Concluding discussion is in Section 5.

2 Model

2.1 Basic set up

In this section, we present the construction and rebalancing of LDI-ES portfolios. We begin in a setting where portfolios are invested in some combination of a risk-free bond with a return of r_f, and a risky asset, such as the stock index, whose returns are normally distributed with an expected return of μ and a standard deviation of σ; i.e., $N(\mu, \sigma^2)$. We denote the interval of each period as h, and the portfolio horizon as T. The number of periods is then $N = T/h$. Each period is indexed by j, where $j \in \{1, 2, \ldots, N\}$. Wealth at the beginning of each period is W_{j-1}, and the target wealth at the end of the horizon is H.

At the beginning of each period j we invest a proportion $w_j \in [0, 1]$ in the risky asset, and $(1 - w_j)$ in the risk-free asset. The portfolio return, r_j, over period j is normally distributed with mean and variance as follows:

$$r_j \sim N\left[w_j \mu h + (1 - w_j) r_f h, w_j^2 \sigma^2 h\right] \tag{1}$$

We can express the *single-period* portfolio return over period j as:

$$r_j = w_j(\mu h + \sigma\sqrt{h} \cdot z) + (1 - w_j) r_f h, \quad z \sim N(0, 1) \tag{2}$$

Now suppose we maintain this configuration from the start of period j through the end of period N. Assuming returns are independent and identically distributed, then the *multi-period* return distribution can be expressed as follows:

$$R_j \sim N\left[(w_j \mu h + (1 - w_j) r_f h)(N - j + 1), w_j^2 \sigma^2 h (N - j + 1)\right] \equiv N[\mu_j, \sigma_j^2] \tag{3}$$

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[4] See `http://www.cato.org/publications/commentary/chiles-social-security-lesson-us`, by Jose Pinera, former Secretary of Social Security in Chile. See also Nuschler (2010).

where $(N - j + 1)$ is the number of periods remaining to the end of the investment horizon T, and we have written the mean and variance of R_j compactly as μ_j and σ_j^2, respectively.

2.2 LDI-ES portfolio construction and rebalancing

Consider a portfolio with wealth W_{j-1} at the beginning of period j. Portfolio wealth at the end of the terminal period N is then described by:

$$W_N = W_{j-1} e^{R_j} \tag{4}$$

where W_N is lognormally distributed since $R_j \sim N[\mu_j, \sigma_j^2]$.

Based on this, an LDI-ES investor can proceed to find the optimal allocation to the risky asset w_j given his target wealth and shortfall tolerance. Specifically, this entails finding the highest value of w_j to maximize the expected value of the portfolio at the horizon, i.e., $E_j[W_N]$, subject to the constraint that the expected shortfall below a set target, H, is less than K.[5] Here $E_j[\cdot]$ stands for the expectation at the beginning of period j, and the expected shortfall constraint of the portfolio for period j is expressed as:

$$ES_j \equiv E_j \left[H - W_N | W_N < H \right] \leq K \tag{5}$$

Or, since H is exogenous, we have

$$ES_j \quad = \quad H - E_j \left[W_N | W_N < H \right], \tag{6}$$

which leaves us to compute $E_j[W_N|W_N < H]$, the expectation of a truncated lognormal variable W_N.

Proposition 1: If $R_j \sim N[\mu_j, \sigma_j^2]$, and $W_N = W_j e^{R_j}$, then

$$E_j[W_N|W_N < H] \quad = \quad W_j \exp\left[\mu_j + \sigma_j \lambda(\alpha) + \frac{1}{2}\sigma_j^2[1 - \delta(\alpha)]\right] \tag{7}$$

$$\alpha \quad = \quad \frac{H_j - \mu_j}{\sigma_j} \tag{8}$$

$$H_j \quad = \quad \ln(H/W_j) \tag{9}$$

$$\lambda(\alpha) \quad = \quad \frac{-\phi(\alpha)}{\Phi(\alpha)} \tag{10}$$

$$\delta(\alpha) \quad = \quad \lambda(\alpha)[\lambda(\alpha) - \alpha] \tag{11}$$

where $\phi(\cdot)$ is the standard normal density function, and $\Phi(\cdot)$ is the standard normal distribution function.

Proof: Because W_N is lognormal, the expectation is

$$E_j[W_N|W_N < H] = W_j \exp\left[E_j(R_j|R_j < H_j) + \frac{1}{2}Var_j(R_j|R_j < H_j)\right].$$

Since R_j is normally distributed, standard calculations for truncated normal random variables give us that $E_j(R_j|R_j < H_j) = \mu_j + \sigma_j\lambda(\alpha)$, and $Var_j(R_j|R_j < H) = \sigma_j^2[1 - \delta(\alpha)]$, where the expressions for $\lambda(\alpha)$ and $\delta(\alpha)$ are given above. See Greene (2003), chapter 22. ■

[5] A solution to the problem where the expected shortfall is managed locally, i.e., only for one a single-period horizon, not out to maturity, is presented in Schulmerich and Trautmann (2003).

3 Portfolio Dynamics and Rebalancing

We examine the dynamic path of portfolio wealth over N periods, given an initial wealth level W_0, a target wealth H, and an expected shortfall allowance K. We optimize period by period, ensuring at each period that we maximize the expected terminal wealth subject to the expected shortfall allowance.

At the beginning of period j, we maximize $E_j[W_N]$ by finding the optimal proportion, w_j, to be allocated to the stock index, subject to the constraint that $E_j[H - W_N|W_N < H] \leq K$ and that $0 \leq w_j \leq 1$. We increase w_j until $E_j[H - W_N|W_N < H] = K$; or, equivalently, until $E_j[1 - W_N/H|W_N/H < 1] = K/H$. We use the latter ratio to present our optimization problem in percentage shortfalls rather than nominal shortfalls.

Once we have the optimal w_j at the beginning of period j, we draw the random one-period return of the stock index, which, combined with the risk-free asset determined our one-period return, r_i, which determines the end-of-period portfolio wealth, $W_j = W_{j-1}e^{r_j}$. We continue in this fashion until the end of period N.

There are times when no feasible solution exists. That is, no feasible w_j exist because wealth at the beginning of period j is insufficient to meet the target wealth, H, given the expected shortfall allowance, K, and the number of periods left in the investment horizon, $N - j + 1$. We consequently compare four methods to restore feasible solutions: 1) infusing cash into the portfolio, 2) increasing the investment horizon, 3) increasing the expected shortfall allowance, or 4) accepting a lower target wealth.

Throughout our numerical examples, we assume that T, the investment horizon, is 20 years, that $h = 1$ so rebalancing occurs once a year, and that the total number of periods is $N = T/h = 20$. We assume an annual risk-free rate of $r_f = 0.03$, and an expected annual return on the stock index of $\mu_j = 0.07$ with a volatility of $\sigma_i = 0.20$. We generate $m = 1,000$ simulated portfolio paths, resulting in a distribution of terminal wealth (W_N).

We now proceed to examine the process of LDI-ES rebalancing for a range of target wealth, H, expected shortfall allowances, K, and the four methods available to ensure feasible solutions.

3.1 LDI-ES portfolio rebalancing

In Table 1 (Panel A), we consider an investor with an initial wealth of $W_0 = \$500,000$ and a target wealth of $H = \$1,000,000$. She sets her shortfall allowance at $K = \$100,000$ at the end of the $N = 20$-year investment horizon. This investor's optimal initial portfolio consists of 17.44% in the stock index, with the remaining 82.56% in the risk-free asset.

Following Table 1, suppose that the realized return during the first period is 0%, such that her wealth at the beginning of the second period remains at $\$500,000$. The optimal allocation to the stock index at the beginning of the second period is then 15.23%. Portfolio rebalancing consists of a reduction in the allocations to the stock index from 17.44% to 15.23% and a corresponding increase in the allocation to the risk-free asset.

Now suppose that the realized return during the second period is 10%. The optimal allocation to the stock index at the beginning of the third period is then 20.18%, and our investor rebalances her portfolio by increasing the allocation to the stock index from 15.23% to 20.18% and a corresponding decrease in the allocation to the risk-free asset.

Table 1: Initial portfolio allocation and subsequent rebalancing for a sample realized return path. Asset weights are determined by the LDI-ES portfolio allocation and are constrained between zero and one, inclusive. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold of $1,000,000 (i.e., $Wealth_0/H = 0.50$), an investment horizon of $N = 20$ years, and an allowed expected shortfall of $K/H = \{10\%, 15\%, 20\%\}$. The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum.

j	realized return	W_{j-1}	W_{j-1}/H	w_j*
		Panel A. $K/H = 10\%$		
1	N/A	$500,000	0.5000	0.1744
2	0%	$500,000	0.5000	0.1523
3	+10%	$550,000	0.5500	0.2018
4	-20%	$440,000	0.4400	infeasible
—	—	—	—	—
		Panel B. $K/H = 20\%$		
1	N/A	$500,000	0.5000	0.4540
2	0%	$500,000	0.5000	0.4439
3	+10%	$550,000	0.5500	0.4875
4	-20%	$440,000	0.4400	0.2831
5	+10%	$484,000	0.4840	0.3698
⋮	⋮	⋮	⋮	⋮
12	⋮	$750,000	0.6452	0.7643
⋮	⋮	⋮	⋮	⋮
19	⋮	$1,000,000	1.0000	1.0000

Next, suppose that the realized return during the third period is a 20% loss, which places our investor in an infeasible region. That is, there is no portfolio with $w_i \in (0, 1)$ such that she can reach her $1,000,000 target wealth with a $100,000 shortfall allowance. Our investor can exit the infeasible region by one of the four methods noted earlier.

In Table 1 (Panel B), we relax the expected shortfall allowance by letting $K = \$200,000$, and we repeat the above realized return path. Under this greater shortfall allowance, no infeasible situation arises and the investor proceeds smoothly to the end of her investment horizon. We see that as the portfolio increases in value throughout the investment horizon, the weights in the risky asset increase.

To generalize these observations, in Figures 1 and 2, we demonstrate how the LDI-ES portfolio allocations evolve over time based on where the realized beginning-of-period wealth, W_{j-1}, falls relative to the desired target H and the allowed expected (percentage) shortfall, K/H. Thus, for any given return path, these plots provide clear prescriptions for investors' optimal allocations/rebalancings at each point in time based on their stated goals or liability directives. We make the following observations based on these figures:

1. Within a given figure (i.e., fixing the investment horizon), we observe that, the optimal

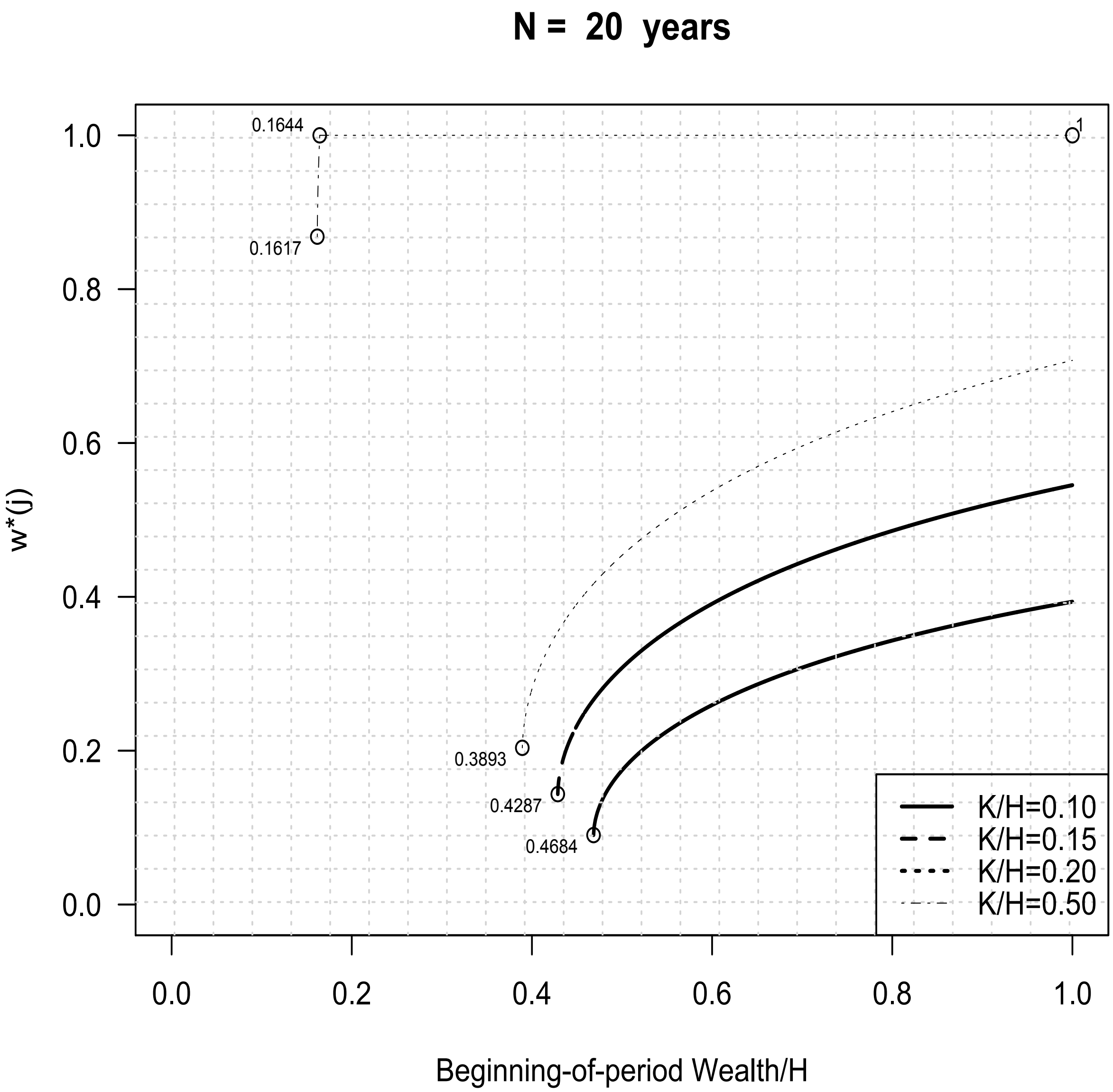

Figure 1: Initial portfolio allocations under LDI-ES for varying $Wealth_{j-1}/H$ ratios, where the allowed expected (percentage) shortfall is $K/H = \{0.10, 0.15, 0.20, 0.50\}$. Asset weights are constrained between zero and one, inclusive. The default setting is an $N = 20$-year investment horizon, with a risk free rate of return of $r = 0.03$, and an expected return on the risky asset of $\mu = 0.07$ and standard deviation $\sigma = 0.20$. As K/H increases the plots show increasing weights w^*.

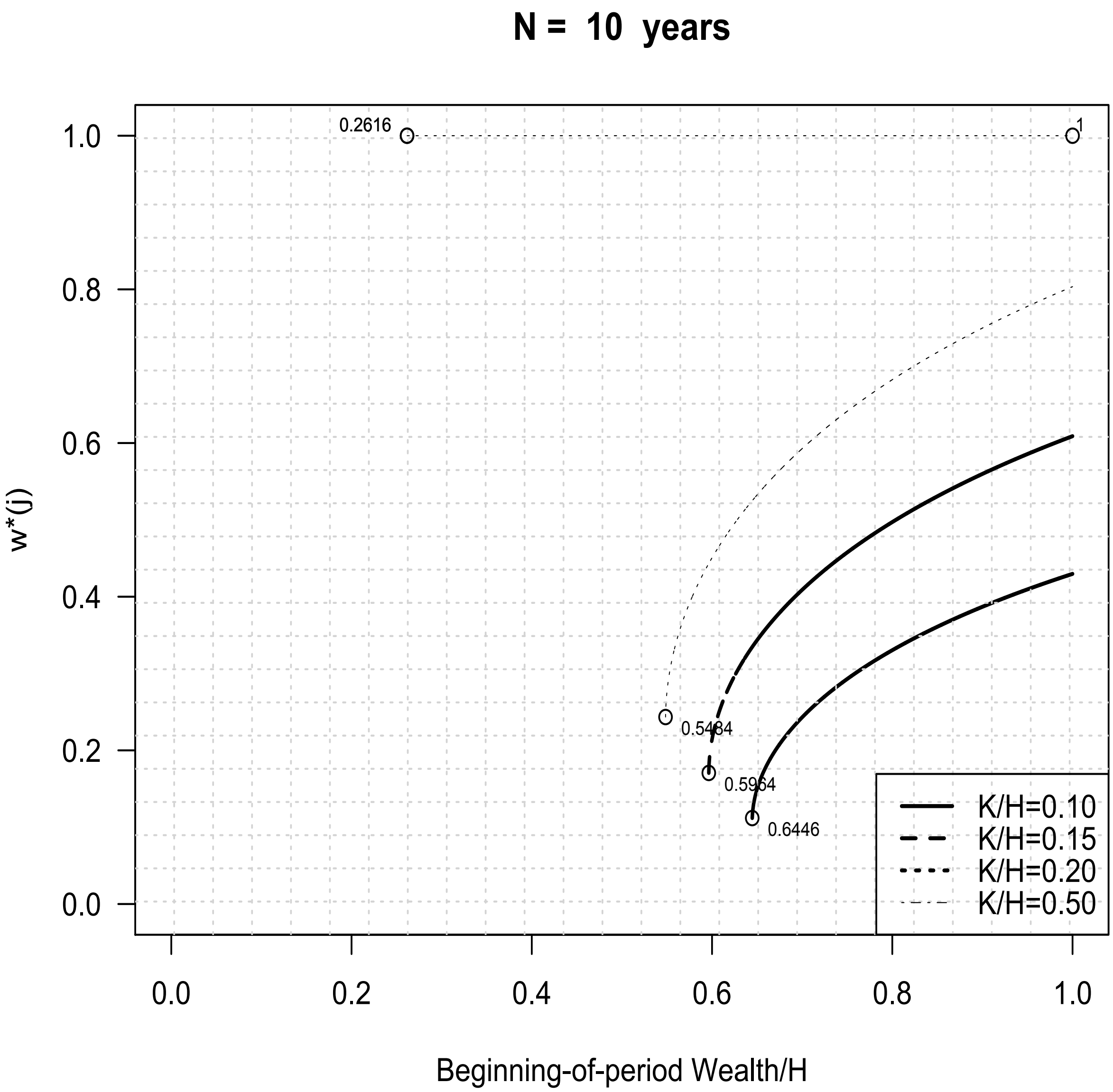

Figure 2: Initial portfolio allocations under LDI-ES for varying $Wealth_{j-1}/H$ ratios, where the allowed expected (percentage) shortfall is $K/H = \{0.10, 0.15, 0.20, 0.50\}$. Asset weights are constrained between zero and one, inclusive. The default setting is an $N = 10$-year investment horizon, with a risk free rate of return of $r = 0.03$, and an expected return on the risky asset of $\mu = 0.07$ and standard deviation $\sigma = 0.20$. As K/H increases the plots show increasing weights w^*.

risky-asset allocation, w_j, increases with K/H, consistent with the notion that K/H represents the amount of shortfall risk investors are willing to accept. That is, for any given W_{j-1}/H ratio, an investor who allows for a greater expected shortfall from her target will find it optimal to allocate a greater proportion of her portfolio to the risky asset.

2. In comparing allocation choices between Figure 1 (where $N = 20$ years) and Figure 2 (where $N = 10$ years), we also observe that the optimal allocation for a given level of $Wealth/H$ differs based on how much time is remaining. In general, w_j increases with $Wealth_{j-1}/H$ (and reaches 100%) far more quickly when there is less time remaining in the investment horizon.

3. We observe no feasible values for w_j at low levels of W_{j-1}/H, and this infeasible region grows for stricter (i.e., lower) values of K/H, since the expected shortfall allowance, $E_j[1 - W_N/H | W_N/H < 1] \leq K/H$, becomes infeasible even at greater W_{j-1}/H ratios when K/H is small. The infeasible region begins at values of $Wealth_{j-1}/H$ lower than the leftmost points of the lines depicting the optimal w_j. For instance, under a 20-year horizon, the minimum feasible W_{j-1}/H is 0.3893 for a $K/H = 0.20$ investor, whereas the minimum feasible W_{j-1}/H is 0.4684 for a $K/H = 0.10$ investor.

4. In comparing the infeasible regions across Figures 1 and 2, we observe that the infeasible region also grows when the time remaining, $N - j + 1$, decreases. For instance, for a $K/H = 0.20$ investor, we see that the minimum feasible W_{j-1}/H is 0.3893 under a 20-year horizon (Figure 1), whereas the the minimum feasible W_{j-1}/H is 0.5484 under a 10-year horizon (Figure 2).

In Table 2, we present the minimum feasible W_{j-1}/H for given K/H and j. For instance, assuming an investment horizon of $N = 20$ periods, an investor standing at time $j = 12$ with an allowed $K/H = 0.15$ will find her original goal infeasible if $W_{j-1}/H < 0.6167$ (see Panel B). That is, with only $N - j + 1 = 9$ years remaining, she cannot achieve an expected shortfall of just 15%, if her realized beginning-of-period wealth is less than 61.67% of her desired target, H. We consider this portfolio *underfunded*.

However, her original goal would still be feasible at this $Wealth/H$ ratio if she had more time left in her investment horizon (e.g., if she were standing back at $j = 1$, or equivalently, if she were willing to increase N; to see this compare Panels A and B of Table 2), or if she were willing to allow a greater expected shortfall (e.g., move up to $K/H = 0.20$, where her current $Wealth/H$ ratio remains within the feasible range). Alternatively, she could return to the feasible range by infusing more wealth (thereby increasing W_{j-1}/H) or by decreasing her target H such that her current wealth is equal to at least 61.67% of her desired threshold (i.e., $W_{11}/H \geq 0.6167$). Thus, underfunding may be resolved by capital infusions, extending the investment horizon, allowing a higher expected shortfall, or reducing the end-of-horizon target wealth.

Overall, Figures 1 & 2, and Tables 1 and 2 provide LDI-ES based prescriptions for portfolio allocations at each point in time based on their stated goals or liability directives. We now proceed to examine in greater detail remedies for infeasibility (i.e., portfolio underfunding).

Table 2: Minimum feasible $Wealth_{j-1}/H$ with corresponding LDI-ES portfolio weight at different points, j, in time for $K/H = \{0.10, 0.15, 0.20, 0.50\}$. The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum. Total investment horizon is $N = 20$ years.

K/H	Min. feasible W_{j-1}/H	corresponding w_j*
Panel A. $j = 1$, $N = 20$		
0.10	0.4684	0.0857
0.15	0.4287	0.1456
0.20	0.3892	0.1987
0.50	0.1617	0.8260
Panel B. $j = 12$, $N = 20$		
0.10	0.6656	0.1143
0.15	0.6167	0.1793
0.20	0.5678	0.2490
0.50	0.2750	0.9982

4 Remedies for Underfunded Portfolios

At any given point j in time, an investor may find that her beginning-of-period wealth W_{j-1} is insufficient to meet her target H under an expected shortfall allowance of K with only $N - j + 1$ periods remaining. We compare four different solutions to portfolio underfunding:

1. **LDI-ES-I**: *Infuse more wealth.* At each point in time j, find the minimum infusion required to bring the portfolio wealth at the beginning of the period (W_{j-1}) back to the minimum feasible level. These infusions come at a cost, i.e., a charge rate r_c. These costs may be the actual cost of funding or the opportunity cost of the money being allocated elsewhere. The terminal wealth then is the value of the portfolio less the repayment of the infusions at cost r_c.

2. **LDI-ES-N**: *Increase investment horizon.* At each point in time, find the minimum increase in N required to bring the portfolio back to a feasible range without the use of wealth infusions.

3. **LDI-ES-K**: *Allow for greater expected shortfall.* At each point in time, find the minimum increase in expected shortfall allowance K that would bring the portfolio back to a feasible range without the use of wealth infusions.

4. **LDI-ES-H**: *Accept a lower terminal wealth target.* At each point in time, find the minimum decrease in target H that would bring the portfolio back to a feasible range without the use of wealth infusions.

Table 3: Sample remedies when portfolio is underfunded. Here, we consider an investor standing at time $j = 4$ with a beginning-of-period wealth equal to 44% of his desired threshold of $1,000,000 (i.e., $Wealth_{j-1}/H = 0.44$). The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum.

Remedy	change	K	H	K/H	N	W_{j-1}	W_{j-1}/H	w_j*
Panel A. Original $K/H = 15\%$, original $N = 20$ years								
do nothing	—	$150,000	$1,000,000	15%	20	$440,000	0.4400	infeasible
increase W_j	$33,000	$150,000	$1,000,000	15%	20	$473,000	0.4730	0.1465
increase N	3	$150,000	$1,000,000	15%	23	$440,000	0.4400	0.2077
increase K	$39,300	$189,300	$1,000,000	18.93%	20	$440,000	0.4400	0.1959
decrease H	$55,100	$150,000	$944,900	15.87%	20	$440,000	0.4657	0.1589
Panel B. Original $K/H = 10\%$, original $N = 20$ years								
do nothing	—	$100,000	$1,000,000	10%	20	$440,000	0.4400	infeasible
increase W_j	$75,300	$100,000	$1,000,000	10%	20	$515,300	0.5153	0.0953
increase N	5	$100,000	$1,000,000	10%	25	$440,000	0.4400	0.0762
increase K	$89,300	$189,300	$1,000,000	18.93%	20	$440,000	0.4400	0.1959
decrease H	$125,500	$100,000	$874,500	11.44%	20	$440,000	0.5031	0.1104

Table 3 (Panel A) illustrates the use of each of these remedies, where we consider an investor standing at time $j = 4$ with beginning-of-period wealth equal to 44% of her desired threshold of $1,000,000 (i.e., $W_{j-1}/H = 0.44$) and an (initial) expected shortfall allowance of $K/H = 15\%$ (i.e., a nominal shortfall allowance of $K = \$150,000$). Under these parameters, this investor's portfolio is underfunded, and her goals are currently infeasible. To continue, this investor could either:

1. Infuse an additional $33,300 into her portfolio, bringing her beginning-of-period wealth to the minimum required $473,000 to meet her LDI goals. Under this scenario, her risky-asset allocation will be $w_j = 0.1465$.

2. Increase her investment horizon by $\Delta N = 3$ years, which brings her back into the feasible realm of meeting her LDI goals (without the use of wealth injections). With more time in her horizon, the investor can afford to take on additional risk and increase the weight in the risky asset. Under this scenario, her risky-asset allocation will be $w_j = 0.2077$.

3. Increase her expected shortfall allowance by $39,300, which brings her new expected (percentage) shortfall allowance to $K/H = 18.93\%$. Under this scenario, her risky-asset allocation will be $w_j = 0.1959$.

4. Decrease her terminal wealth target by $55,100, which brings her current wealth as a percentage of her threshold up to $W_{j-1}/H = 46.57\%$ and increases her expected (percentage) shortfall allowance to $K/H = 15.87\%$. Under this scenario, her risky-asset allocation will be $w_j = 0.1589$.

Table 3 (Panel B) shows the same remedies but for a stricter percentage shortfall allowance of $K/H = 0.10$. Here we see that the required remedies are more drastic. The required infusion increases to \$75,300 (from \$33,000 when $K/H = 0.20$). The required extension of the time horizon is now 5 years, in comparison to the 3 years in Panel A. Alternatively, the expected shortfall allowance must be relaxed by an additional \$50,000 over that in Panel A to \$89,300, or the target H must be be reduced by \$125,500, versus just \$55,100 under the higher expected shortfall allowance.

Table 4: Future value of expected infusions (FVEI) under LDI-ES, compounded at an infusion cost of $r_c = 3\%$ per annum. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold (i.e., $W_0/H = 0.50$, with H normalized to 1). The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum.

K/H	Expected FVEI	[Stdev]
Panel A. Full sample		
0.10	0.0368	[0.055]
0.15	0.0405	[0.073]
0.20	0.0449	[0.087]
0.50	0.0019	[0.018]
Panel B. Ending wealth in bottom 10%		
0.10	0.1375	[0.070]
0.15	0.1794	[0.099]
0.20	0.2212	[0.118]
0.50	0.0177	[0.054]
Panel C. Ending wealth in top 10%		
0.10	0.0055	[0.018]
0.15	0.0032	[0.016]
0.20	0.0007	[0.006]
0.50	0.0000	[0.000]

For a general sense of the total infusions required under different expected shortfall allowances, in Tables 4 and 5, we present the future value of all expected infusions (FVEI) required throughout the investment horizon, compounded at an annual infusion cost of $r_c = \{0.03, 0.07\}$. That is, for each of $K/H = \{0.10, 0.15, 0.20, 0.50\}$, we calculate three different FVEI, taking the average required infusions: 1) across all $m = 1,000$ simulations, 2) for the subset of simulations where terminal wealth is in the bottom 10%, and 3) for the simulations where terminal wealth is in the top 10%. Terminal wealth is calculated after deducting infusion costs at rate r_c, thereby making the sorting of outcomes into deciles less sensitive to infusion costs. We make the following observations:

1. The difference between average infusions required in the unlucky paths (i.e., the bottom 10% of outcomes) and average infusions overall is larger than that between the lucky paths (i.e., the top 10% of outcomes) and average infusions overall. Thus the required infusion distribution is right-skewed, i.e., outliers are large, meaning the strategy often needs large infusions, or equivalently, underfunding is fat-tailed.

Table 5: Future value of expected infusions (FVEI) under LDI-ES, compounded at an infusion cost of $r_c = 7\%$ per annum. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold (i.e., $W_0/H = 0.50$, with H normalized to 1). The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum.

K/H	Expected FVEI	[Stdev]
	Panel A. Full sample	
0.10	0.0597	[0.093]
0.15	0.0660	[0.122]
0.20	0.0731	[0.146]
0.50	0.0030	[0.028]
	Panel B. Ending wealth in bottom 10%	
0.10	0.2668	[0.096]
0.15	0.3459	[0.141]
0.20	0.4122	[0.178]
0.50	0.0293	[0.085]
	Panel C. Ending wealth in top 10%	
0.10	0.0098	[0.034]
0.15	0.0047	[0.029]
0.20	0.0010	[0.008]
0.50	0.0000	[0.000]

2. Expected infusions increase as the shortfall allowance K/H increases, until the shortfall allowance becomes very high (i.e., a slack shortfall constraint), and then expected infusions become very low again, because meeting the shortfall allowance is relatively easy. Thus, we see an inverted u-shape for expected infusions as a function of K/H.

In Table 6, we report analogous statistics on expected increases in investment horizon N. Think of this as the case where an investor is forced to retire later than she wishes to. For instance, an investor with an expected shortfall allowance of $K/H = 0.10$ can expect to increase her investment horizon by $\Delta N = 1.48$ years (with a standard deviation of 1.92) to resolve portfolio underfunding without the use of wealth infusions, though an unlucky $K/H = 0.10$ investor (captured by the simulations where terminal wealth is in the bottom 10%) is expected to increase her investment horizon by $\Delta N = 4.72$ years (with a standard deviation of 2.22). On the other hand, an investor with an allowance of $K/H = 0.20$ can expect to increase her investment horizon by $\Delta N = 1.63$ years (with a standard deviation of 2.88), and an unlucky $K/H = 0.20$ investor is expected to increase her investment horizon by $\Delta N = 7.27$ years (with a standard deviation of 3.46). On the other hand, in the case of $K/H = 0.50$, where the investor is willing to accept a large shortfall, the expected increase in portfolio horizon, even in the worst case bottom decile, is less than a year ($\Delta N = 0.81$ years).

In general, we observe that the expected infusions/changes required are generally greater for a $K/H = \{0.15, 0.20\}$-investor than for a $K/H = 0.10$ investor, suggesting that the

Table 6: Expected increases in investment horizon N required following the LDI-ES portfolio allocation with an allowed expected shortfall of $K/H = \{0.10, 0.15, 0.20, 0.50\}$, where asset weights are constrained between zero and one, inclusive. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold (i.e., $W_0/H = 0.50$), and an original investment horizon of $N = 20$ years. At each point in time j, we calculate the minimum increase in N required to bring the portfolio back to a feasible range without the use of wealth injections. We then take the average required changes: 1) across $m = 1,000$ simulations, 2) across the simulations where terminal wealth is in the bottom 10%, and 3) across the simulations where terminal wealth is in the top 10%. The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum.

K/H	Expected N	Mean change	[Stdev]
Panel A. Full sample			
0.10	21.48	1.48	[1.92]
0.15	21.48	1.48	[2.37]
0.20	21.63	1.63	[2.88]
0.50	20.09	0.09	[0.75]
Panel B. Ending wealth in bottom 10%			
0.10	24.72	4.72	[2.22]
0.15	25.72	5.72	[2.96]
0.20	27.27	7.27	[3.46]
0.50	20.81	0.81	[2.22]
Panel C. Ending wealth in top 10%			
0.10	20.13	0.13	[0.42]
0.15	20.04	0.04	[0.20]
0.20	20.02	0.02	[0.14]
0.50	20	0	[0]

riskier asset allocations undertaken by an investor allowing for greater expected shortfalls require her to apply a portfolio-underfunding remedy more often. In a similar vein, for an investor with a very high expected shortfall allowance (e.g., $K/H = 0.50$), we observe relatively negligible expected changes throughout, since his greater tolerance for shortfall risk enables him to maintain low wealth levels and still remain within reach of his original, albeit laxer, LDI goals.

4.1 Comparison of alternatives

We now compare the terminal W_N distribution from following the 60/40 fixed-proportion strategy, versus using the LDI-ES-I portfolio rebalancing strategy, which takes the requisite infusions when faced with portfolio underfunding. Note that the FP scheme rebalances to the fixed 60/40 equity-debt proportion every year, and is a naive scheme with no shortfall constraint. In calculating each terminal wealth W_N, we subtract any infusions taken at prior points in the path, compounded at an annual infusion cost r_c.

One approach to making a comparison across various schemes for an LDI-ES investor is to convert portfolio outcomes to a single utility number. The LDI-ES investor cares about maximizing terminal wealth and keeping expected shortfall to a constrained level. Hence, we assume a utility function of the LDI-ES investor that is equal to $E(W_T) - c \cdot ES$, where c is the disutility incurred by each unit of shortfall, and we calculate c^0, the break even level of c that makes the LDI-ES and FP cases equal in utility.[6]

That is, we want to know whether the net utility of the expected outcome under LDI-ES investing exceeds that of the expected outcome under FP investing:

$$\begin{aligned} E[W_T]_{LDI-ES} - c \cdot ES_{LDI-ES} &> E[W_T]_{FP} - c \cdot ES_{FP} \\ \iff E[W_T]_{LDI-ES} - E[W_T]_{FP} &> c \cdot (ES_{LDI-ES} - ES_{FP}) \end{aligned} \tag{12}$$

Thus, we see that for positive values of $(ES_{LDI-ES} - ES_{FP})$, Equation 12 is satisfied if $c < \frac{E[W_T]_{LDI-ES} - E[W_T]_{FP}}{ES_{LDI-ES} - ES_{FP}} \equiv c^0$. In other words, when $ES_{LDI-ES} - ES_{FP} > 0$, LDI-ES investing is more likely to be the dominant strategy for *greater* values of c^0. On the other hand, we see that for negative values of $(ES_{LDI-ES} - ES_{FP})$, Equation 12 is satisfied when $c > c^0$. That is, when $ES_{LDI-ES} - ES_{FP} < 0$, LDI-ES investing is more likely to be the dominant strategy for *lesser* values of c^0.

4.1.1 The LDI-ES-I case

Table 7 presents the W_N distribution parameters based on $m = 1,000$ simulations and annual infusion costs of $r_c = \{0.03, 0.07, 0.14\}$. As before, we consider an investor who has $N = 20$ periods in her investment horizon, a starting wealth equal to 50% of her desired target, and expected shortfall allowances of $K/H = \{0.10, 0.15, 0.20, 0.50\}$.

At $r_c = 0.03$ (Panel B.1) the LDI-ES-I strategy dominates the FP 60/40 scheme when allowable shortfall is high. For example, if the allowable shortfall is $K/H = 0.20$, then we must have $c < 13.335$ for LDI-ES to be preferred to FP, and this is almost surely the case. When $K/H = 0.50$, again we must have $c < 6.684$ for LDI-ES to be preferred. However, when shortfall allowance is low, i.e., $K/H = 0.10$, then the condition is reversed, whereby we must now have $c > 4.800$ for LDI-ES to be preferred, i.e., FP is more likely to be preferred unless, the cost of shortfall is high. A similar conclusion is drawn when $K/H = 0.15$, since here, we must have $c > 3.244$ for LDI-ES to be preferred. Overall, we observe that under a lower infusion cost, LDI-ES investing is generally preferred to FP investing.

At greater infusion costs, $r_c = \{0.07, 0.14\}$, greater values of c^0 consistently signify a greater range of investors who prefer LDI-ES-I, since investors who have a penalty cost (i.e., a coefficient that corresponds to shortfall risk tolerance) $c < c^0$ prefer LDI-ES-I over FP. If c^0 is very low, as is the case for high infusion cosrs, then only investors who are not concerned with shortfall will use LDI-ES-I.

We also observe from Table 7 that LDI-ES-I has much higher positive skewness than FP. This arises from truncating the left tail in the former scheme but not in the latter.

[6] In Ang, Chen, and Sundaresan (2013) the utility function is stated in returns and they also discount disutility from variance in returns. In contrast, we express utility and shortfall in terms of terminal wealth but since our calculations are normalized by target H, they are relative outcomes. Furthermore, we do not include a haircut for variance risk, as LDI-ES investors only care about shortfall, and aim to maximize wealth otherwise.

Table 7: Distribution parameters of terminal wealth, W_N, from following the LDI-ES portfolio allocations each period versus a 60/40 fixed proportion, where asset weights are constrained between zero and one, inclusive. At each point in time j, we calculate the minimum infusion required to top off the portfolio if the realized wealth at the beginning of the period is below the minimum feasible W_{j-1}/H. We then subtract out these infusions from the ending wealth at a cost of $r_c = \{3\%, 7\%, 14\%\}$ per annum. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold (i.e., $W_0/H = 0.50$, with H normalized to 1). Expected return, standard deviation, skewness, kurtosis, probability of shortfall, and expected shortfall (given that the threshold is not met) are estimated based on $m = 1,000$ simulations. The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum. Assuming a utility function of the LDI-ES investor that is equal to $E(W_T) - c \cdot ES$, where c is the risk aversion towards shortfall, we report c^0, the break even level of c that makes the LDI-ES and FP cases equal in utility. Note that c^0 is equal to the ratio of difference in expected W_T across the two schemes divided by the difference in ES across LDI-ES and FP. In the last column we report the range of c relative to c^0 where LDI-ES does better than FP.

Method	K/H	Mean	Stdev	Skew	Pr(shortfall)	ES	c^0
Panel A. 60/40 FP, no infusions							
60/40 FP	—	1.7007	0.9372	94.64	20.90%	0.2471	
Panel B.1. LDI-ES, with infusion cost of $r_c = 3\%$ per annum							
LDI-ES	0.10	1.2721	0.6158	131.13	42.10%	0.1578	> 4.800
LDI-ES	0.15	1.6222	1.1753	161.37	33.30%	0.2229	> 3.244
LDI-ES	0.20	2.0274	1.8278	188.12	28.90%	0.2716	< 13.335
LDI-ES	0.50	2.8738	3.1995	196.13	27.50%	0.4226	< 6.684
Panel B.2. LDI-ES, with infusion cost of $r_c = 7\%$ per annum							
LDI-ES	0.10	1.2124	0.6575	110.67	45.10%	0.2585	< -42.833
LDI-ES	0.15	1.5561	1.2241	145.18	35.90%	0.3616	< -1.263
LDI-ES	0.20	1.9543	1.8817	174.42	31.30%	0.4484	< 1.260
LDI-ES	0.50	2.8708	3.2019	195.72	27.50%	0.4334	< 6.281
Panel B.3. LDI-ES, with infusion cost of $r_c = 14\%$ per annum							
LDI-ES	0.10	1.0561	0.8179	42.37	49.60%	0.5214	< -2.350
LDI-ES	0.15	1.3820	1.4051	88.39	38.90%	0.7412	< -0.645
LDI-ES	0.20	1.7615	2.0758	126.34	34.30%	0.9223	< 0.090
LDI-ES	0.50	2.8637	3.2088	194.47	27.50%	0.4593	< 5.481

However, as Panels B.2 and B.3 of Table 7 show, if the infusion cost is high, then FP is likely to be preferred by more investors, unless the shortfall allowance is sufficiently high. More generally, these results indicate that low risk-taking investors may be better off with simple FP rebalancing, whereas investors willing to take on more risk are better off with LDI-ES rebalancing.

4.1.2 The LDI-ES-N/K/H cases

We also observe that expected terminal wealth and standard deviations of terminal wealth increase with the expected shortfall allowance, K/H, and we make similar observations in Table 8, where we compare terminal wealth distributions across the other remedies for portfolio underfunding.

With regard to the break even values c^0 that compare the LDI-ES-N, LDI-ES-K, and LDI-ES-H schemes with FP rebalancing, we observe that the three LDI-ES schemes are usually preferred by investors except when shortfall allowance is low ($K/H = 0.10$). To see this assume that $c = 3$ in which case for shortfall allowances in the set $\{0.15, 0.20, 0.50\}$, LDI-ES is preferred to FP. That is, we see that in most cases, the values of c^0 are such that a large range of investors would prefer the LDI-ES schemes over the FP one. It is usually when the utility penalty for shortfall (c) is high and the shortfall constraint K/H is tight, that FP is preferred to LDI-ES.

In Table 8, we explore the other three (non-infusion) variants of the LDI-ES scheme, i.e., extending maturity (number of periods N), increasing shortfall allowance K, or lowering the promised target H. In addition to reporting c^0 (which measures investor preference for LDI-ES over FP), the rightmost column in Table 8 reports c^1. For investors with shortfall aversion coefficient $c < c^1$ the LDI-ES-I scheme is preferred to the other three schemes. We compare these alternative schemes with LDI-ES-I at an infusion cost of 3%, biasing results in favor of finding the infusion case to be preferable. We see that LDI-ES-I is never preferred to the LDI-ES-K or LDI-ES-H schemes, since $c^1 < 0$ in all cases but one, meaning that only shortfall risk seeking investors would prefer LDI-ES-I. Even in the LDI-ES-N scheme, the values of c^1 are extremely small positive values, suggesting that most investors would prefer to extend their target horizon rather than provide infusions; it appears that delaying gratification is preferred to paying to avoid postponing it.

At higher infusion costs, the performance of LDI-ES-I would be even worse. A comparison of the mean and standard deviation of terminal wealth appears to be similar across across the three alternate remedies when compared to the infusion case. Hence, accounting for mean and variance, the expected shortfall is worse in the infusion case. Thus, the other three remedies appear to be superior, and suggest that infusions should only be a last resort if other approaches are not permissible (for example, when it is not possible to push back retirement age, or reduce retirement benefits that may be legally stipulated).

Overall, the relationships across schemes is not quantifiably transitive under the c metric. For example, LDI-ES-I (at infusion cost $r_c = 0.03$) has a range of investors who prefer it to FP (Table 7). We also see that the other three schemes LDI-ES-N/K/H are preferred by a wide range of investors to LDI-ES-I (Table 8). These three schemes are also mostly preferred to FP as well, *but* by a smaller range of investors than those who prefer these three schemes over LDI-ES-I.

Table 8: Terminal wealth W_N distribution parameters from following the LDI-ES portfolio allocations, where asset weights are constrained between zero and one, inclusive. Portfolio underfunding is now resolved by either increasing the investment horizon N (Panel A), increasing the expected shortfall allowance K (Panel B), or decreasing the desired target wealth H (Panel B). For instance, at each point in time j, we calculate the minimum decrease in H required to bring an underfunded portfolio back into the feasible domain. Here, we consider an investor with a starting wealth equal to 50% of his desired threshold (i.e., $W_0/H = 0.50$, with H normalized to 1). Expected return, standard deviation, skewness, kurtosis, probability of shortfall, and expected shortfall (given that the threshold is not met) are estimated based on $m = 1,000$ simulations. The expected return on the risky asset each period is $\mu_j = 0.07$, with a standard deviation of $\sigma_j = 0.20$, and the risk-free rate is $r_f = 0.03$ per annum. Assuming a utility function of the LDI-ES investor that is equal to $E(W_T) - c \cdot ES$, where c is the risk aversion towards shortfall, we report c^1, the break even level of c that makes the LDI-ES with infusion remedy and alternative remedy cases (ΔN, ΔK, and ΔH) equal in utility. Note that c^1 is equal to the ratio of difference in expected W_T across the two schemes divided by the difference in ES across LDI-ES (with infusion) and an alternate LDI-ES scheme. We compare the alternate remedies with the infusion case when the cost of infusion is $r_c = 0.03$. In the second-last column we report the range of c relative to c^0 where LDI-ES-N/K/H does better than FP. In the last column we report the range of c relative to c^1 where LDI-ES-I does better than LDI-ES-N/K/H.

K/H	Mean	Stdev	Skew	Pr(shortfall)	ES	c^0	c^1
			Panel A. LDI-ES, allowing increases in N				
0.10	1.2663	0.6012	140.55	42.20%	0.1377	> 3.971	< 0.289
0.15	1.6111	1.1688	165.36	35.40%	0.1941	> 1.691	< 0.385
0.20	2.0130	1.8262	189.84	31.10%	0.2458	> -240.231	< 0.558
0.50	2.8735	3.1996	196.10	27.50%	0.4235	< 6.649	< -0.333
			Panel B. LDI-ES, allowing increases in K				
0.10	1.3141	0.6381	123.63	38.60%	0.1542	> 4.161	< -11.667
0.15	1.6538	1.1784	159.20	32.00%	0.2065	> 1.155	< -1.927
0.20	2.0460	1.8272	187.60	28.90%	0.2580	< 31.679	< -1.368
0.50	2.8739	3.1993	196.15	27.50%	0.4220	< 6.708	< -0.167
			Panel C. LDI-ES, allowing decreases in H				
0.10	1.2764	0.6124	132.90	41.90%	0.1491	> 4.330	< -0.494
0.15	1.6276	1.1704	163.14	33.20%	0.2062	> 1.787	< -0.323
0.20	2.0312	1.8228	189.59	29.10%	0.2500	< 113.966	< -0.176
0.50	2.8739	3.1994	196.14	27.50%	0.4223	< 6.696	< -0.333

5 Concluding Discussion

In this paper, we analyze a rebalancing scheme based on behavioral portfolio theory investors who manage downside risk while maximizing the terminal wealth in their portfolios in a multi-period setting. Investors overweight risky assets when their portfolio wealth and shortfall tolerance are high. We find that unless investors have very stringent shortfall risk thresholds, LDI-ES rebalancing does better for them than fixed proportion rebalancing. We considered critical issues of underfunding, where there is no portfolio that can meet the shortfall constraint, and remedial action is needed. We find that portfolio infusions are not as effective in resolving underfunded situations as other measures such as increasing risk, cutting back on target liabilities/goals, and extending portfolio horizon.

There are many extensions and applications that we envisage to follow up on the results in this paper. We may consider gradual infusions, i.e., rather than making the full required infusion at the time of underfunding, we inject only partial infusions and live with some degree of underfunding. Such an approach may prove optimal especially when the cost of underfunding r_c is high. Another interesting question that arises is whether these remedies for underfunded portfolios change when we do not have a single target liability on one date T, but a stream of liabilities to meet on dates $T_1, T_2, \ldots, T_N$. Is it sufficient to treat each liability on separate dates as a distinct instance of the problem handled here in this paper, or are these interacting problems that need a different rebalancing approach? While we have looked at each of the four approaches to handling underfunding, we have assumed they are "all-or-nothing" strategies, and are implemented in a mutually exclusive manner. Finding the optimal "mixed-remedy-strategy" is a complicated problem that needs separate analysis.

References

Akcay, Y., and Yalcin, A. (2010). Optimal Portfolio Selection with a Shortfall Probability Constraint: Evidence from Alternative Distribution Functions, *Journal of Financial Research* 33(1), 77–102.

Amenc, Noel., Lionel Martellini, Felix Goltz, and Vincent Milhau (2010). New Frontiers in Benchmarking and Liability-Driven Investing, Working paper, EDHEC-Risk Institute.

Ang, Andrew., Bingxu Chen, and Suresh Sundaresan (2013). Liability Driven Investment with Downside Risk, *Journal of Portfolio Management* 40(1), 71–87.

Basak, S., and A. Shapiro (2001). Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices, *Review of Financial Studies* 14, 371–405.

Basak, S., A. Shapiro, and L. Tepla (2006). Risk Management with Benchmarking, *Management Science* 52, 542–557.

Binsbergen, Jules van., and Michael Brandt (2009). Optimal Asset Allocation in Asset Liability Management, Working paper, Stanford University.

Browne, Sid. (2000). Risk-Constrained Dynamic Active Portfolio Management, *Management Science* 46(9), 1188–1199.

Das, Sanjiv., and Meir Statman (2013). Options and Structured Products in Behavioral Portfolios, *Journal of Economic Dynamics and Control* 37(1), 137–153.

Follmer, Hans., and Peter Leukert (1999). Quantile Hedging, *Finance and Stochastics* 3(3), 251–273.

Greene, William. (2003). *Econometric Analysis*, 5th edition, Prentice-Hall, NJ.

Levy, H., and M. Sarnat (1972). Safety First: An Expected Utility Principal, *Journal of Financial and Quantitative Analysis* 7, 1829–1834.

Martellini, L., and V. Milhau (2009), Measuring the Benefits of Dynamic Asset Allocation Strategies in the Presence of Liability Constraints, Working paper, EDHEC-Risk Institute.

Martellini, L., and V. Ziemann, (2010). Improved estimates of higher-order comoments and implications for portfolio selection. *Review of Financial Studies* 23(4), 1467–502.

Novy-Marx, R., and J. Rauh (2011). Public Pension Promises: How Big Are They and What Are They Worth? *Journal of Finance* 66(4), 1211–1249.

Nuschler, D. (2010). Social Security Reform: Current Issues and Legislation, *Congressional Research Service*, September, Report for Congress: RL33544.

Roy, A. (1952). Safety First and the Holding of Assets, *Econometrica* 20, 431-449.

Sharpe, William., and Larry Tint (1990). Liabilities - A New Approach, *Journal of Portfolio Management*, Winter, 5–10.

Shefrin, Hersh., and Meir Statman (2000). Behavioral portfolio theory, *Journal of Financial and Quantitative Analysis* 35(2), 127–151.

Schulmerich, M., and S. Trautmann (2003). Local Expected Shortfall-Hedging in Discrete Time, *European Finance Review* 7, 75–102.